MURAKKAB ARALASH TURDAGI UCHINCHI TARTIBLI TENGLAMA UCHUN CHEGARAVIY MASALA.

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Qobiljon Solijonovich G'oziyev
Fizika-matematika fanlari nomzodi,dotsent,Farg'ona,O'zbekiston.
e-mail:ksgaziyev1965@gmail.com
Nishonova Shahnozaxon Toxirjon qizi
Fizika-matematika fanlari bo'yicha falsafa doktori(PhD),Farg'ona,O'zbekiston.
e-mail:shahnozanishonova910@gmail.com
Boltaboyeva Sarvinozxon Yoqubjon qizi
Farg'*ona davlat universiteti 2-kurs magistranti, Farg'ona, O'zbekiston.*
e-mail:sarvinozyoqubjonovna@gmail.com

Annotatsiya

Mazkur *maqolada chegaralanmagan sohada uchinchi tartibli tenglama uchun chegaraviy masala qaralgan bo'lib,masalaning yechimining yagonaligi integral energiya usulida,yechimining mavjudligi esa Fredgolmning ikkinchi tur integral tenglamalari nazariyasidan foydalangan holda isbotlangan.*

Kalit so`zlar

Yuqori tartibli tenglama, Fredgolm integral tenglamasi, yechimning yagonaligi, yechimning mavjudligi, Grin funksiyasi.

Murakkab va murakkab-aralash tipdagi tenglamalar uchun ustozlarimiz M.S.Salohiddinov $[2]$, T.D.Jo'rayev $[1]$ hamda ularning shogirdlari tomonidan chegaraviy masalalar qo'yilib, ushbu masalalarni o'rganish nazariyalari yaratilgan. Shulardan biri $[4]$ da chegaralangan sohada umumiy tenglama uchun chegaraviy masalalar tahlil etilgan. Bu maqolada uchinchi tartibli aralash turdagi tenglama uchun chegaralanmagan sohada chegaraviy masala ko'rilgan.

Masalaning qo'yilishi:

CHegaralanmagan $D=\{D_1 \cup D_2 \cup y=0\}$ sohada

$$\begin{cases} \frac{\partial}{\partial y}(U_{xx}+U_{yy})+C(x,y)U(x,y)=0, y>0 \\ \frac{\partial}{\partial y}(U_{xx}-U_{yy})=0, y<0 \end{cases}$$

(1)

tenglamani qaraymiz.

Bunda $D_1=\{(x,y): x>0, y>0\}$, $D_2=\{(x,y): x>0, y>-x\}$, $C(x,y)$ berilgan funksiya.

Masala. $\bar{D}$ sohada uzluksiz va (1) tenglamani xamda quyidagi chegaraviy shartlarni qanoatlantiruvchi $U(x,y)$ funksiya topilsin:

$$U(0,y)=\varphi_1(y),\ 0\le y<\infty,$$

(2)

$$U(x,-x)=\psi_1(x),\ 0\le x<\infty,$$

(3)

$$\frac{\partial U(x,-x)}{\partial n}=\psi_2(x),\ 0<x<\infty,$$

(4)

$$\lim_{R\to\infty}U_y=0, R^2=x^2+y^2, x>0, y>0.$$

(5)

Bu yerda $\varphi_i(y), \psi_i(x),\ (i=1,2)$ berilgan funksiyalar bo'lib, $\varphi_1(0)=\psi_1(0)$ kelishuv shartini qanoatlantiradi. (1) tenglamada quyidagi

$$\frac{\partial}{\partial y}U=V \qquad (6)$$

belgilash kiritsak,u holda tenglama

$$\begin{cases} \Delta V+CU=0 \\ \square V=0 \end{cases} \qquad (7)$$

ko'rinishga keladi.

Bu yerda
$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-elliptik\ \ operator$$
$$\square=\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-giperbolik\ \ operator$$

(6) belgilashga asosan (2)-(5) chegaraviy shartlarga asosan $V(x,y)$ funksiya uchun quyidagi chegaraviy shartlarni olamiz:

$$V(x,y)|_{y=-x}=\frac{1}{2}\left(\psi_1'(x)+\sqrt{2}\psi_2(x)\right)=\psi(x)$$

(8)

$$V(x,y)|_{x=0}=\varphi_1'(y)$$

(9)

$$\lim_{R\to\infty}V(x,y)=0$$

(10)

Masala yechimining yagonaligi.

1-teorema. Agar (1) – (5) masala yechimi mavjud bo`lsin va

a) $|C(x,y)|\le\frac{N_1}{R}$, $R\to\infty$, $N_1=const$, $C_x(x,y)\ge 0$.

Agar yuqoridagi shartlar bajarilsa, u holda masala yechimi yagona bo'ladi.

Isbot: Qo'yilgan masala yechimining yagonaligini integral energiya usuldan foydalanib isbotlaymiz. Shuning uchun

$$\psi_i(x)=\varphi_1(y)=0\ (i=1.2)$$

(11)

bo'lsin. U holda (6) belgilashga ko'ra

$$\begin{cases}\Delta V+CU=0\\ \Box V=0\end{cases}$$

(7)

tenglamaga ega bo'lamiz. Chegaraviy shartlar esa

$$V(x,y)|_{y=-x}=0$$

(12)

$$V(x,y)|_{x=0}=0$$

(13)

ko'rinishida bo'ladi.

Ushbu sohani qaraylik: $D_{1R}=\{(x,y):x^2+y^2<R^2,x>0,y>0\}$ D_{2R} esa $AC:x+y=0$ va $BC:x-y=R$ chiziqlar bilan chegaralangan soha. D_{1R} sohaning chegarasini $\sigma_R=\{(x,y):x^2+y^2=R^2,x>0,y>0\}$, $A(0,0)B(R,0)$ va $D(0,R)$ $A(0,0)$ chiziqlar bilan belgilaymiz.(7) ning birinchi lokal operator qatnashgan tenglamasini $V(x,y)$ funksiyaga ko'paytirib, D_{1R} soha bo'yicha integral olamiz:

$$\iint_{D_{1R}}V\left[(V_{xx}+V_{yy})+CU\right]dxdy=0$$

(14)

Yuqoridagi (14) tenglikni quyidagi ko'rinishda yozib olamiz:

$$\iint_{D_{1R}} \left[(VV_x)_x + (VV_y)_y - (V_x)^2 - (V_y)^2 + \frac{1}{2}(CU^2)_x - \frac{1}{2}C_x U^2 \right] dxdy = 0 \qquad (15)$$

Bu tenglikka Gauss-Ostrogradskiy formulasini qollab,

$$\int_{\partial D_{1R}} V\left[V_x \cos(n,x) + V_y \cos(n,y)\right] ds + \frac{1}{2}\int_{\partial D_{1R}} CU^2 dy = \iint_{D_{1R}} \left[V_x^2 + V_y^2 + C_x U^2\right] dxdy$$

ko'rinishidagi ifodaga kelamiz. Bir jinsli (13) shartdan hamda AB chiziq ustiga $dy = 0$ ekanligidan foydalansak, quyidagiga ega bo'lamiz.

$$\int_{G_R} V\frac{\partial V}{\partial n} - \int_{AB} V(x,0)V_y(x,0)dx + \frac{1}{2}\int_{G_R} CU^2 dy = \iint_{D_{1R}} \left[V_x^2 + V_y^2 + C_x U^2\right] dxdy \qquad (16)$$

Endi (7) ning ikkinchi tenglamasini $V(x,y)$ ga ko'paytirib, D_{2R} soha bo'yicha integrallaymiz hamda ushbu tenglikni quyidagi ko'rinishda yozib olamiz:

$$\iint_{D_{2R}} (VV_x)_x - (VV_y)_y - (V_x)^2 + (V_y)^2 dxdy = 0 \qquad (18)$$

(18) tenglikka Gauss-Ostrogradskiy formulasini qo'llasak ushbu ifodani olamiz.

$$\int_{AC} VV_x dy + VV_y dx + \int_{CB} VV_x dy + VV_y dx + \int_{BA} VV_x dy + VV_y dx = \iint_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy \, (19)$$

Bir jinsli (12) va (13) shartlardan hamda AC chiziq ustida $dx = -dy$, CB chiziq ustida $dx = dy$ ekanligidan foydalanib (19) tenglikning chap tomonini hisoblaymiz.

$$\int_{CB} VV_x dy + VV_y dx = \int_{CB} V\left(V_x dy + V_y dx\right) + \int_{CB} V\left(V_x dx + V_y dy\right) = \frac{1}{2}V^2(B)$$

Hosil bo'lgan natijani (19) ga qo'ysak

$$\int_{BA} VV_y dx + \frac{1}{2}V^2(B) = \iint_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy$$

(20)

tenglik hosil bo'ladi. $\xi = x - y,\ \eta = x + y$ belgilashlarga ko'ra $x = \frac{\xi + \eta}{2},\ y = \frac{\eta - \xi}{2},\ V_x = V_\xi + V_\eta,\ V_y = -V_\xi + V_\eta$ tengliklar o'rinli bo'ladi.

Bunga asosan (20) dan

$$\iint\limits_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy = \frac{1}{4} \iint\limits_{D'_{2R}} \left[(V_\xi + V_\eta)^2 - (-V_\xi + V_\eta)^2\right] d\xi d\eta =$$

$$= \frac{1}{4} \iint\limits_{D'_{2R}} 4V_\xi V_\eta d\xi d\eta = \iint\limits_{D'_{2R}} \frac{\partial}{\partial \eta}(VV_\xi) d\xi d\eta -$$

$$-\iint\limits_{D'_{2R}} VV_{\xi\eta} d\xi d\eta = \iint\limits_{D'_{2R}} \frac{\partial}{\partial \eta}(VV_\xi) d\xi d\eta = -\int\limits_{A_1C_1+C_1B_1+B_1A_1} VV_\xi d\xi$$

C_1B_1 chiziqda $d\xi = 0$ ekanligini e'tiborga olsak

$$\iint\limits_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy = -\int\limits_{A_1C_1+B_1A_1} VV_\xi d\xi$$

tenglikka ega boʻlamiz. (12) va (13) shartlardan foydalanamiz hamda tenglikni oʻng tomonini hisoblaymiz.

U holda ushbu

$$\int\limits_{D_{2R}} \left[V_x^2 - V_y^2\right] dxdy = -\frac{1}{2} V^2(B) \qquad (21)$$

tenglik oʻrinli. Quyidagi

$V(x,0) = \tau_1(x)$

$V_y(x,0) = \nu_1(x)$

belgilashni kiritsak,(20) ni chap tomonini integrallab,

$$\int\limits_{BA} VV_y = \int\limits_B^A \tau_1(x)\nu_1(x)dx = \frac{1}{2}V^2(B) - \frac{1}{2}V^2(B) = 0$$

ekanligini hisoblash mumkin.

Bizga

$$\int\limits_A^B \tau_1(x)\nu_1(x)dx = 0$$

ekanligi ma'lum.(10) ga asosan $R \to \infty$da limitga oʻtamiz. Bundan esa

$$\int\limits_0^\infty \tau_1(x)\nu_1(x)dx = 0 \qquad (22)$$

tenglik hosil boʻladi. (22) ni (16)ga qoʻyamiz va $R \to \infty$ da limitga oʻtamiz. U holda teorema shartiga va (13) ga asosan

$$\iint\limits_{D_1} \left[V_x^2 + V_y^2\right] dxdy + \iint\limits_{D_1} C_x U^2 dxdy = 0$$

tenglik oʻrinli boʻladi.Agar $C_x = 0$ boʻlsa, soʻnggi tenglikdan $V_x = 0$, $V_y = 0$ ekanligi kelib chiqadi. Bundan esa $V = const$, (13) shartga koʻra $V = U_x = 0$ ga ega boʻlamiz. Bu tenglamani $U(0, y) = 0$ shart ostida yechsak, $\bar{D}_1$sohada $U \equiv 0$ekanligi kelib chiqadi.

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Yuqoridagi tenglikdan D_1 sohada $U \equiv 0$ bo'ladi. Funksiyaning yopiq sohada uzluksizligidan $\bar{D}_1$ sohada $U \equiv 0$ ayniyatni olamiz. $\bar{D}_1$ sohada $V \equiv 0$ ekanligidan $\tau_1 \equiv 0$ tenglikka ega bo'lamiz. D_2 sohada (7) ning ikkinchi tenglamasi uchun Koshi masalasining yechimi yagonaligidan $V \equiv 0$ ekanligi kelib chiqadi.

$V = U_x = 0$

tenglamani $U(x,-x) = 0$ shart bilan yechsak, $\bar{D}_2$ da $U \equiv 0$ trivial yechimga ega ekanligini ko'rsatish mumkin.

Yechimning yagonaligi isbotlandi.

Masala yechimining mavjudligi.

2-teorema. Agar 1-teorema shartlari bajarilsa va

a) $\psi_i(y), \varphi(x)$ $(i = 1,2)$ funksiyalar uzluksiz;

b) $|\varphi_1(y)| \le \frac{C_2}{R^2}$, $C(x,y) \le \frac{C_3}{R^{2+\varepsilon}}$,

$|\psi_1(x)| \le \frac{C_4}{R^{\varepsilon}}$, $|\psi_2(x)| \le \frac{C_5}{R^{\varepsilon}}$, $R \to \infty$

$\left(c_j = conts,\ j = \overline{1,5}\right)$ shartlarni qanoatlantirsa, u holda $(1)-(5)$ masalani yechimi mavjud.

Isbot: (6) belgilashga ko'ra (1) tenglamadan D_1 sohada

$$V_{xx} + V_{yy} = -CU \tag{23}$$

tenglamani, (2), (5), shartlardan esa

$$V(0,y) = \varphi_1'(y) \tag{9}$$

$$\lim_{R\to\infty} V(x,y) = 0 \tag{10}$$

shartlarni olamiz. D_2 sohada (1) tenglamani

$$U_{xx} - U_{yy} = \omega(x) \tag{24}$$

ko'rinishida yozib olamiz, bunda $\omega(x)$ noma'lum funksiya. (24) ning umumiy yechimi $[3]$

$$U(x,y) = F(x+y) + \Phi(x-y) + \frac{1}{4}\int_0^y d\eta \int_{x-y+\eta}^{x+y-\eta} \omega(\xi)\, d\xi \tag{25}$$

ko'rinishida bo'ladi. (4) shartdan foydalanib, (25) dan

$$\omega(x) = \sqrt{2}\psi_2(-y) \tag{26}$$

ekanligini topamiz. (4) shart va (26) ni nazarda tutsak quyidagiga ega bo‘lamiz:

$$U(x,y)=F(x+y)+\psi_1\left(\frac{x-y}{2}\right)-F(0)+P(x,y)$$

(27)

bu yerda

$$P(x,y)=\frac{\sqrt{2}}{4}\int_0^y dy\int_{x-y+\eta}^{x+y-\eta}\psi_2(-\eta)d\eta$$

(27) dan va $U(x,0)=\tau(x)$, $U_y(x,0)=\nu(x)$ belgilashlarga ko‘ra

$$\tau'(x)-\nu(x)=\psi_0(x) \qquad (28)$$

AB dagi noma’lum funksiyalar uchun birinchi munosabatni olamiz.

Bu yerda

$$\psi_0(x)=-\psi_1'\left(\frac{x}{2}\right)+P_x(x,0)-P_y(x,0)$$

(28)

dan x bo‘yicha xosila olib

$$\tau''(x)-\nu'(x)=\psi_0'$$

(29)

ko‘rinishidagi munosabatga ega bo‘lamiz.

Endi D_1 sohada (23) tenglamaning (9) va $V_y(x,0)=\nu'(x)$ shartlarni qanoatlantiruvchi yechimini Grin funksiyasi orqali yozamiz (7):

$$V(x,y)=\int_0^\infty G(x,y,\xi,0)\nu'(\xi)d\xi-\int_0^\infty G_\xi(x,y,0,\eta)\varphi_1'(\eta)d\eta+$$
$$+\iint_{D_1} G(x,y,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta \qquad (30)$$

bunda

$$G(x,y,\xi,\eta)=\frac{1}{2\pi}\ln\left|\frac{z^2-\bar{\zeta}^2}{z^2-\zeta^2}\right|,\quad z=x+iy,\ \zeta=\xi+i\eta$$

(30) da $y=0$ deb x bo`yicha xosila olamiz.

$$V_x(x,0)=-\frac{1}{\pi}\int_0^\infty\left[\frac{1}{\xi-x}+\frac{1}{\xi+x}\right]\nu'(\xi)d\xi+$$
$$+\frac{2}{\pi}\int_0^\infty\frac{\eta^2-x^2}{(x^2+\eta^2)^2}\varphi_1'(\eta)d\eta+\iint_{D_1}G_x(x,0,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta$$

$V_x(x,0)=\tau''(x)$ ekanligini hisobga olsak va (29) ga ko`ra

$$\nu'(x)+\frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\xi-x}+\frac{1}{x+\xi}\right]\nu'(\xi)d\xi-\iint_{D_1}G_x(x,0,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta=F_1(x)$$

(31)

Bu yerda

$$F_1(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\eta^2-x^2}{(x^2+\eta^2)^2}\varphi_1'(\eta)d\eta-\psi_0'\ (x)$$

(31) tenglamaga ushbu

$$Kf=f(x)+\frac{1}{\pi}\int_0^{\infty}\left[\frac{1}{\xi-x}-\frac{1}{\xi+x}\right]f(\xi)d\xi$$

operatorni ta'sir ettirsak, unga ekvivalent bo'lgan (8)

$$\frac{1}{\pi}\int_0^{\infty}\frac{\nu'(\xi)}{\xi-x}=F_2(x)$$

(32)

tenglamani olamiz. Bunda

$$F_2(x)=\frac{1}{2}K\bar{F}_1(x)\,,$$

$$\bar{F}_1(x)=F_1(x)-\iint_{D_1}G_x(x,0,\xi,\eta)C(x,y)U(x,y)d\xi d\eta$$

(32) tenglamani $x=0$da birdan kichik maxsuslikka ega va $x\to\infty$da chegaralangan yechimi [5]

$$\nu'(x)=-\frac{1}{\pi\sqrt{x}}\int_0^{\infty}\sqrt{\xi}\frac{F_2(\xi)}{\xi-x}d\xi$$

(33)

ko'rinishda ifodalanadi.Endi (30) formuladan foydalanib, (6) tenglamani (2) shart ostida yechamiz.

$$U=\int_0^{y}\left[\int_0^{\infty}\left(\frac{t}{t^2+(y-\eta)^2}+\frac{t}{t^2+(y+\eta)^2}\right)\varphi_1'(\eta)d\eta-\right.$$

$$\left.-\frac{1}{2\pi}\int_0^{\infty}\ln\left|\frac{(t+\xi)^2+y^2}{(t-\xi)^2+y^2}\right|\nu'(\xi)d\xi+\iint_{D_1}G(t,y,\xi,\eta)C(\xi,\eta)U(\xi,\eta)d\xi d\eta\right]dt+\varphi_1(y)$$

U holda quyidagi tenglamaga kelamiz.

$$U(x,y)=\iint_{D_1}K(x,y,\xi,\eta)U(\xi,\eta)d\xi d\eta+\int_0^{\infty}K_2\nu'(\xi)d\xi=F_2(x,y)$$

(34)

Bunda

$$K(x,y,\xi,\eta)=C(\xi,\eta)\int_0^y G(x,t,\xi,\eta)dt\ ,$$

$$K_2(x,y,\xi)=(x+\xi)\ln\left|(x+\xi)^2+y^2\right|-(x-\xi)\ln\left|(x-\xi)^2+y^2\right|-4x+2y\,arctg\frac{x+\xi}{2}-$$
$$-2y\,arctg\frac{x-\xi}{2}-2\xi\ln\left|\xi^2+y^2\right|+4\xi\ ,$$

$$F_2(x,y)=\int_0^\infty K_1(x,y,\eta)\varphi_1'(\eta)d\eta+\varphi_1(y)\ ,$$

$$K_1(x,y,\eta)=\frac{1}{2}\ln\left|\frac{x^2+(y+\eta)^2}{(y+\eta)^2}\right|+\frac{1}{2}\ln\left|\frac{x^2+(y-\eta)^2}{(y-\eta)^2}\right|$$

(34) tenglamaga $\nu'(x)$ ning (33) dagi ifodasini qo‘yamiz.

$$U(x,y)-\iint_{D_1}K_4(x,y,\xi,\eta)U(\xi,\eta)d\xi d\eta=F_{20}(x,y)$$

(35)

Bunda ushbu tenglama hosil bo‘ladi.

$$K_4(x,y,\xi,\eta)=K(x,y,\xi,\eta)+\int_0^\infty K_2(x,y,\xi_1)K_3(\xi_1,\xi,\eta)d\xi_1\ ,$$

$$F_{20}(x,y)=F_2(x,y)-\int_0^\infty K_2(x,y,\xi)F_0(\xi)d\xi,$$

$$K_3(x,\xi,\eta)=\frac{1}{\sqrt{x}}\int_0^\infty G_x(\xi_1,0,\xi,\eta)d\xi_1-\int_0^\infty\frac{\sqrt{\xi_1}}{\xi_1-x}d\xi_1\int_0^\infty\left[\frac{1}{\bar{\xi}-\xi_1}+\frac{1}{\bar{\xi}+\xi_1}\right]G_x(\bar{\xi},0,\xi,\eta)d\bar{\xi}\ ,$$

$$F_0(x)=-\frac{1}{\pi\sqrt{x}}\int_0^\infty\frac{\sqrt{\xi_1}F_1(\xi_1)}{\xi_1-x}d\xi_1+\frac{1}{\pi\sqrt{x}}\int_0^\infty\frac{\sqrt{\xi_1}}{\xi_1-x}d\xi_1\int_0^\infty\frac{1}{\bar{\xi}-\xi_1}+\frac{1}{\bar{\xi}+\xi_1}\Big]F_1(\bar{\xi})d\bar{\xi}$$

(35) tenglama Fredgolm II tur tenglamasi bo‘lib, yechimning mavjudligi yagonalik teoremasidan kelib chiqadi. Mavjudlik teoremasi shartlari bajarilganda $U(x,y)$ funksiya masalaning barcha shartlarini qanoatlantiradi.

Yechim mavjudligi isbotlandi.

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