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EXPLICIT MULTIDIMENSIONAL INTERPOLATION AND ITS ERROR

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Abstract

The article discusses explicit formulas for multidimensional chaotic interpolation: generalized interpolation formulas of the Newton and Lagrange types. These formulas are generally irrational, but among them, there are polynomial formulas of even order. Estimates of the remainder terms of the generalized Newton and Lagrange interpolation formulas are obtained, as well as for arbitrary interpolation formulas, based solely on the smoothness of the interpolated function and the vanishing of the remainder at the grid points.

Keywords

multidimensional chaotic interpolation, main explicit interpolation formulas, remainder term of the interpolation formula, dependence of the remainder term on the smoothness of the interpolated function and on the interpolation formula.

Аннотация

В статье рассматриваются явные формулы многомерной хаотической интерполяции: обобщённые интерполяционные формулы типа Ньютона и Лагранжа. Формулы эти в общем иррациональные, но среды них есть и полиномиальные формулы четного порядка. Получены оценки остаточных членов обобщённых интерполяционных формул Ньютона и Лагранжа, и произвольных интерполяционных формул только на основе гладкости интерполируемой функции и обращения в нуль остатка в узлах сетки точек.

Ключевые слова

интерполяционная многомерная хаотическая интерполяция, основные явные интерполяционные формулы, остаточный член интерполяционной формулы, зависимость остаточного члена от гладкости интерполируемой функции и от интерполяционной формулы.

Аннотация

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Мақолада ихтёрий нуқталарда кўп ўзгарувчили ошкор интерполяция формулалари қаралган. Улар классик Ньютон ва Лагранж формулаларининг умумлашмасидир. Қолдиқ ҳаднинг функция синфидан ва интерполяция формуласидан боғлиқлиги.

Калит сўзлар

кўп ўзгарувчили хаотик ошкор интерполяция, умумлашган интерполяция формулалари. Қолдиқ ҳаднинг функция синфидан боғлиқлиги.

1. Problem Statement. The problem of chaotic multidimensional interpolation is posed as follows [1-6]. Let the norm $|x| = \sqrt{(x,x)} = \sqrt{(x^1)^2 + ... + (x^m)^2}$ of points $x = (x^1,...,x^m)$ are introduced in the Euclidean space R^m, where a finite domain $\Omega = \{x = (x^1,...,x^m)\} \in R^m$ and the grid of chaotic points (nodes) $\Delta_n = \{x_i, i = 0,...,n\}$, as well as the values $y_i = f(x_i), i = 0,...,n$ of some continuously smooth function $y = f(x)$, are given . It is required to determine an interpolation formula (surface) of possibly simple structure,

$$I_n(x) = I_n(f,x) = \sum_{i=0}^{n} c_i \varphi_i(x), \; I_n(x_i) = f(x_i), i = 0,..., n . \quad (1)$$

Such an interpolation formula (surface) is represented as follows:

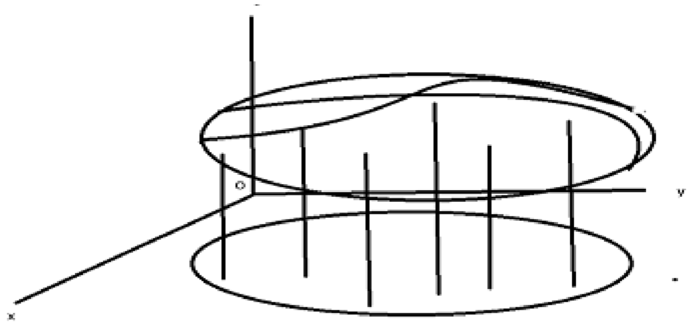

Fig. 1. Graphical representation of the interpolation surface.

The difference

$$R_n(x) = f(x) - I_n(x), R_n(x_i) = 0, i = 0..n \quad (2)$$

is called the error or residual term of the interpolation formula.

In the article, $I_n(x)$ will be constructed, and the general form of $R_n(x)$ will be determined.

In one dimensional case we have following Newton and Lagrange formulas:

$$N_n(x) = f(x_0) + f[x_0,x_1](x - x_0) + \cdots f[x_0, ..., x_n](x - x_0) ... (x - x_n),$$

$$L_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x), \quad l_i(x) = \prod_{j\neq i}^{n} \frac{x - x_j}{x_i - x_j}, h_i = |x_i - x_{i-1}|,$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, ..., f[x_0, ..., x_k] = \frac{f[x_0,...,x_{k-2},x_k] - f[x_1,...,x_{k-1},x_k]}{x_k - x_{k-1}},$$

$R_n(x) = f(x) - N_n(x) = f[x_0, \dots, x_n, x](x - x_0) \dots (x - x_n)$,

$R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x - x_0) \dots (x - x_n)$.

2. Explicit Multidimensional Interpolation Formulas.

The overview of multidimensional chaotic interpolation first appeared in the works [1], [2]. We propose the following new explicit generalized interpolation formula of the Newton and Lagrange type [6]:

$N_n(x) = f(x_0) + \sum_{k=1}^{n} f[x_0, \dots, x_k]\omega_k^{\alpha}(x), \omega_k^{\alpha}(x) = |x - x_0|^{\alpha} \dots |x - x_{k-1}|^{\alpha}$ (3)

$L_n(x) = \sum_{i=0}^{n} f(x_i)\varphi_i(x), \varphi_i(x) = \sum_{j \neq i}^{n} |x - x_j|^{\alpha}|x_i - x_j|^{-\alpha}$, (4)

$f[x_0, \dots, x_k] = (f[x_0, \dots, x_{k-2}, x_k] - f[x_0, \dots, x_{k-1}])/|x_k - x_{k-1}|^{-\alpha}$.

In the following lemma, for simplicity, let us assume that $\alpha = 1$. In (3,4), the polynomial case is obtained for even α, for example, when $\alpha = 2$. Let us define the unknown coefficients $a_i, i = 1 \dots n$, from the interpolation conditions: $N_n(x_i) = N_i(x_i) = f(x_i)$. Similarly, the following holds true for the one-dimensional case:

Lemma 1. The following equalities hold: $a_i = f[x_0, \dots, x_i], i = 1 \dots n$.

Theorem 1. The following multidimensional multi-point expansion holds:

$$f(x) = N_n(x) + R_n(x), \quad (5)$$

where $N_n(x)$ is determined by formula (3), and the remainder is given as

$R_n(N_n, x) = f(x) - N_n(x) = \omega_n^{\alpha}(x)f[x_0, \dots, x_n, x]$. (6)

Proof. For definitions of generalized divided differences we have

$f(x) = f(x_0) + f[x_0, x]|x - x_0|^{\alpha}$, $f[x_0, x] = f[x_0, x_1] + f[x_0, x_1, x]||x - x_1|^{\alpha}|, \dots, f[x_0, \dots x_{n-1}, x] = f[x_0, \dots, x_n] + f[x_0, \dots x_n, x]|x - x_n|^{\alpha}$.

Then we inside each new divided differences in old and we get (5).

Corollary 1. For the function $u(x)$: $u(x_i) = 0, i = 0..n$ the following formula holds:

$u(x) = \omega_n^{\alpha}(x)u[x_0, \dots, x_n, x]$ (7)

Theorem 2. Formulas (3) and (4) are interpolation formulas.

Indeed, the relations $N_n(x_k) = f(x_k), k = 0..n$, can be proven in two ways.

1) It is evident that $R_n(x_i) = 0 \rightarrow N_n(x_i) = f(x_i), i = 0..n$.

2) We calculate directly. For example, it is evident that

$N_n(x_0) = f(x_0), N_n(x_1) = f(x_0) + f[x_0, x_1]|x_1 - x_0|^{\alpha} = f(x_0), \dots,$

$$N_n(x_k) = N_k(x_k) + \sum_{i=k}^{n-1} \omega_i(x_k) f[x_0, \dots, x_{i+1}] = N_k(x_k) = f(x_k), k = 1..n$$

The properties of the basic functions, such as $\varphi_i(x_j) = \delta_{ij} = 1, i = j; \delta_{ij} = 0, i \neq j$ ensure the fulfillment of the interpolation conditions. Let $h_i = |x_i - x_{i-1}|$.

The interpolation formulas in rectangle grid with error consider in [7].

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3. Representation for Divided Differences.

For divided differences, we will find a representation for the case: $\alpha = 1$.

Theorem 3. Let $f(x) \in C^n[\Omega]$. Then there exist point ξ_n and directions $e_1,...,e_n$ such that the following equalities hold:

$$f[x_0,\dots,x_n] = D_{e_n\dots e_1} f(\xi_n), \alpha = 1,$$

$$f[x_0,\dots,x_n] = h_1^{\alpha-1} \dots h_n^{\alpha-1} D_{e_n\dots e_1} f(\xi_n), \alpha \geq 1 . \qquad (8)$$

Proof. We apply the method of induction. Steps $k = 1, 2$. By definition, we have:

$$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{h_1} = \frac{(f'(\xi_1),(x_1-x_0))}{h_1} = D_{e_1} f(\xi_1), \dots,$$

$$f[x_0,\dots,x_2] = D_{e_2} f\left[x_0,\dot{\xi}_2\right] = D_{e_2 e_1} f(\xi_2).$$

Step $k-1$. Suppose the following representation holds:

$$f[x_0,\dots,x_{k-1}] = D_{e_{k-1}} \dots D_{e_1} f[x_0,\dots,x_{k-1}]$$

Step k. Using the definition and the assumption from the previous step, we have:

$$f[x_0,\dots,x_k] = D_{e_k} f\left[x_0,\dots,x_{k-2},\dot{\xi}_k\right] = D_{e_k} \dots D_{e_1} f(\xi_k)$$

Now we representations of divided differences for α:

$$f[x_0,x_1] = (f(x_1) - f(x_0))/h_1^{\alpha} = \frac{h_1^{\alpha-1}(f(x_1)-f(x_0))}{h_1} = h_1^{\alpha-1} D_{e_1} f(\xi_1),$$

$$f[x_0,x_1,x_2] = h_1^{\alpha-1} h_2^{\alpha-1} D_{e_2 e_1} f(\xi_2), \dots,$$

$$f[x_0,\dots,x_n] = h_1^{\alpha-1} \dots h_n^{\alpha-1} D_{e_n\dots e_1} f(\xi_n), h_i = |x_i - x_{i-1}|, i = 1..n.$$

4. Estimate of the Remainder Term in Multidimensional Explicit Interpolation.

There exists Bézout's theorem, which states that a polynomial $P_n(x)$ that vanishes at a point: $x = x_0 \colon P_n(x_0) = 0$ is divisible by: $x - x_0$: $P_n(x) = (x - x_0)P_{n-1}(x)$. Bézout's theorem can be generalized for a continuous function $f(x)$ instead of the first derivative: $f'(x)$. In this case, the formula of Lagrange's finite increments holds true [8]:

$$f(x) = f(x_0) + f'(\xi)(x - x_0), \xi = x_0 + \theta(x - x_0). \qquad (9)$$

For $f(x_0) = 0$, this implies an equality that generalizes Bézout's theorem for a function of one independent variable:

$$f(x) = f'(\xi)(x - x_0) = g_0(x)(x - x_0) = \left(f' \circ \xi_0(x)\right)(x - x_0). \qquad (10)$$

Let us generalize Bézout's theorem for a function of several variables. Consider a function that has a set of zeros: $f(x) \in C^{m+1}[\Omega], \mathrm{f}(x_i) = 0, \mathrm{i} = 0..\mathrm{n}$.

Lemma. If $f(x) \in C^1[\Omega], f(x_0) = 0$, then the (10) equality holds with:

$$f'(x) = grad f(x) = [f_{x_i}, i = 1..n]^T.$$

Theorem 4. (Theorem on Condensing Zeros). If $f(x) \in C^{n+1}[\Omega], f(x_i) = 0, i = 0..n$, then there exists a bilinear form (the differential of order $n+1$) $g_n(x): R^{m+1} \to R$ such that

$$f(x) = g_n(x)\big((x - x_0) \dots (x - x_n)\big) = \big(g_n(x), \big((x - x_0) \dots (x - x_n)\big)\big). \quad (11)$$

Proof. By Lagrange's finite increment formula, we have (10).

Next, using the equalities $f(x_1) = \cdots f(x_n) = 0$ sequentially, we have:

$$f(x_1) = g_0(x_1)(x_1 - x_0) = 0 \Rightarrow g_0(x_1) = 0 \equiv g_0(x) = g_1(x)(x - x_1) \Rightarrow$$
$$f(x) = g_1(x)(x - x_0)(x - x_1)$$

Continuing this process, we have:

$$f(x) = g_0(x)(x - x_0) = g_1(x)(x - x_0)(x - x_1) = g_n(x)(x - x_0) \dots (x - x_n)$$

It remains to determine the form of the function $g_n(x)$. It is clear that

$$g_0(x) = f_0'\big(x_1 + \theta_1(x - x_1)\big) = \big(f_0' \circ \xi_0(x)\big), \xi_0(x) = x_0 + \theta_0(x - x_0),$$
$$g_1(x) = g_0'\big(x_1 + \theta_1(x - x_1)\big) = \big(g_0' \circ \xi_1(x)\big), \xi_1(x) = x_1 + \theta_1(x - x_1), \dots,$$
$$g_n(x) = g_{n-1}'\big(x_n + \theta_n(x - x_n)\big) = \big(f_0' \circ \xi_n(x)\big), \xi_n(x) = x_n + \theta_n(x - x_n).$$

It is clear that

$$g_n(x) = g_{n-1}' \circ \xi_n(x) = g_{n-2}' \circ \xi_{n-1} \circ \xi_n(x) = f^{(n+1)} \xi_0 \circ\circ\circ \xi_n(x) = f^{(n+1)}(\xi(x)).$$

Corollary 1. If $f(x) \in C^{n+1}[\Omega], f(x_i) = 0, i = 0..n$, and α=1 then there exists a point $\xi(x) \in \Omega$ such that we have:

$$f(x) = D_{e_n \dots e_1} f(\xi)(x - x_0) \dots (x - x_n), \alpha=1. \quad (12)$$

Corollary 2. If $f(x) \in C^{n+1}[\Omega], f(x_i) = 0, i = 0..n$, and α≠1 then there exists a point $\xi(x) \in \Omega$ such that we have:

$$f(x) = h_1^{\alpha-1} \dots h_n^{\alpha-1} D_{e_n \dots e_1} f(\xi)(x - x_0) \dots (x - x_n) \quad (12')$$

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